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Relativistic space-charge-limited bipolar flow

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Relativistic generalization of the Child-Langmuir law is derived for flows involving two oppositely charged species. A large enhancement of the space-charge-limited current in the ultrarelativistic limit, in which both species are accelerated to energies exceeding their rest-mass energies, is demonstrated. [S1063-651X(98)01807-8]

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The subject of the present Brief Report is the generalization of the Child-Lamgmuir law [1,2] to the case in which the current is carried by two species of opposite electric charges, accelerated to relativistic energies. This case is relevant to many astrophysical plasmas as well as to the future, GeV range, ion diodes. For example, in the astrophysical context, the motional electric field in the pulsar magnetospheres is expected to give rise to potential differences as large as 10^{11} V along the magnetic field lines [3]; similarly, in plasmas accreting on magnetized white dwarfs, electrostatic potentials differences can arise that can accelerate particles to energies much greater than proton rest-mass energy [4]. Such large potential differences induce field-aligned currents that may be carried by relativistic electrons and positrons, as in the pulsar case, or by relativistic electrons and ions as in the accretion case. The simplest estimate of the magnitude of such currents is given by the one-dimensional approximation of Child-Langmuir theory.

Thus let us consider a steady-state, space-charge-limited flow in a planar diode, with electrodes perpendicular to the x direction and separated by distance d. For the case when the current is carried by only one nonrelativistic species, the relation between the current density J and the diode voltage U is given by the classical Child-Langmuir law, which can be conveniently expressed in the form

$$J=\frac{2^{3/2}}{9}\alpha^{1/2}\frac{cU}{2\pi d^2}, \tag{1}$$

where $\alpha=qU/mc^2\ll 1$, and where q and m denote the particle charge and mass, respectively.

The relativistic single-species flow in such a configuration was analyzed by Jory and Trivelpiece [5] who found that, in the ultrarelativistic regime, the Child-Langmuir law takes the form

$$J=\frac{cU}{2\pi d^2}. \tag{2}$$

When the current is carried by two species of opposite charge, the electrostatic potential Φ is governed by the Poisson equation

$$\frac{d^2\Phi}{dx^2}=4\pi(q_-n_- - q_+n_+), \tag{3}$$

where n is the particle density and q is the charge; here and in the following, subscripts + and − refer to positive and negative particles, which are emitted from the cathode ($\Phi=0$) and the anode ($\Phi=U$), respectively.

Particle densities can be expressed in terms of particle currents $J_\pm$ and flow velocities $u_\pm$ for each particle species:

$$J_\pm=q_\pm n_\pm u_\pm . \tag{4}$$

In the notation adopted here, all currents and velocities are positive and the total current $J=J_+ + J_-$. The particle motion is described in the cold two-fluid theory. Because of particle number conservation, both J_+ and J_- are constant. Relativistic energy conservation implies that

$$u_+=c\{1-[1+\alpha_+(1-\xi)]^{-2}\}^{1/2} \tag{5}$$

and

$$u_-=c\{1-[1+\alpha_-\xi]^{-2}\}^{1/2}, \tag{6}$$

where $\alpha_\pm=q_\pm U/m_\pm c^2$, $\xi=\Phi/U$, and m stands for particle mass.

With the aid of Eqs. (4)–(6), the Poisson law (3) can be integrated once to yield

$$\left(\frac{d\xi}{dx}\right)^2=\frac{8\pi}{cU}\left\{\frac{J_+}{\alpha_+}\sqrt{[1+\alpha_+(1-\xi)]^2-1}+\frac{J_-}{\alpha_-}\sqrt{[1+\alpha_-\xi]^2-1}-C\right\}, \tag{7}$$

where C is an integration constant. The diode current is at the space-charge limit when the electric field vanishes on both electrodes, that is, when $d\xi/dx=0$ for $\xi=0$ and $\xi=1$. From this, it follows that

$$\frac{J_+}{\alpha_+}\sqrt{(1+\alpha_+)^2-1}=\frac{J_-}{\alpha_-}\sqrt{(1+\alpha_-)^2-1}=C. \tag{8}$$

Observe that $J_+\approx J_-$ in the ultrarelativistic limit ($\alpha_\pm\gg 1$), independently of the mass and charge ratios of involved species. This is in contrast to the nonrelativistic flow ($\alpha_\pm\ll 1$) for which $J_-/J_+=\sqrt{\mu}$ where $\mu\equiv\alpha_-/\alpha_+=m_+q_-/m_-q_+$.

Inserting Eq. (8) into Eq. (7), and performing the integration, for the total current one finally obtains

$$J=F(\alpha_+,\alpha_-)\frac{cU}{2\pi d^2}, \tag{9}$$

where

$$F(\alpha_+,\alpha_-)=\frac{1}{4}\left[\left(1+\frac{2}{\alpha_+}\right)^{-1/2}+\left(1+\frac{2}{\alpha_-}\right)^{-1/2}\right] \times\left\{\int_0^1 d\xi\left[\sqrt{(1-\xi)^2+\frac{2\xi(1-\xi)}{\alpha_++2}} + \sqrt{\xi^2+\frac{2\xi(1-\xi)}{\alpha_-+2}}-1\right]^{-1/2}\right\}^2. \tag{10}$$

In general, the function F has to be obtained by numerical integration. As an example, a result of such an integration for the electron-proton flow case is shown in Fig. 1. We find simplifying expressions in the following asymptotic regimes.

(1) Nonrelativistic regime, $\alpha_\pm \ll 1$:

$$J=0.586(1+\mu^{-1/2})\alpha_-^{1/2}\frac{cU}{2\pi d^2}. \tag{11}$$

(2) Intermediate regime (for $\mu \gg 1$), $\alpha_+ \ll 1 \ll \alpha_-$:

$$J=\frac{\pi^2}{4}\frac{cU}{2\pi d^2}. \tag{12}$$

(3) Ultrarelativistic regime, $\alpha_\pm \gg 1$:

$$J=2(1+\mu^{-1/2})^{-2}\alpha_+\frac{cU}{2\pi d^2}. \tag{13}$$

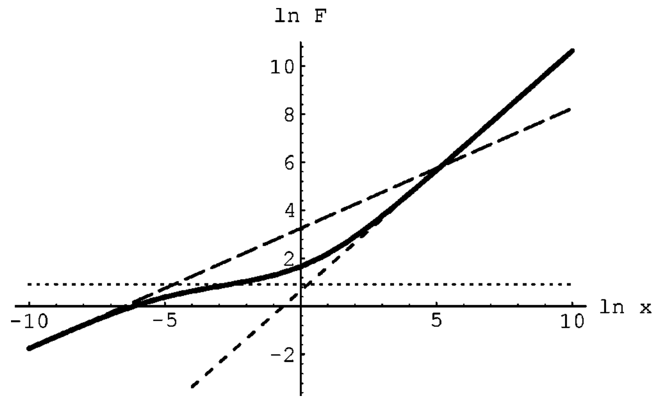

FIG. 1. The numerically computed dependence (solid line) of $\ln F(x,\mu x)$ on $\ln x$ for the electron-proton flow ($\mu=1840$); long- and short-dashed lines show nonrelativistic and ultrarelativistic asymptotes, given by Eqs. (11) and (13), respectively; the intermediate regime [Eq. (12)] is shown by the dotted line.

These asymptotic formulas are compared with the numerical solution in Fig. 1.

Observe that, in the ultrarelativistic regime, the functional dependence of current on voltage in the two species case ($J \sim U^2$) is qualitatively different from that for the single-species case ($J \sim U$); this is in contrast to the nonrelativistic and intermediate regimes in which the current, while enhanced by a factor on the order of unity [6], has the same functional dependence on voltage in both cases. This additional enhancement of the space-charge-limited current in the ultrarelativistic regime is due to the fact that, to the lowest order in $1/\alpha$, densities of both current-carrying species are equal; thus the current can be increased without a corresponding increase in the space charge.

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